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ANGULAR POWER SPECTRUMS AND THEIR RELATIONSHIP TO AUTOCORRELATION FUNCTIONS OF APERTURE ANTENNAS

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SUMMARY

Normalized angular power spectrum representations have been derived for linear and circular aperture antennas in terms of their autocorrelation functions. Examples of their use are given.

INTRODUCTION

In communication theory, the correlation function and its properties are well-known and invaluable in analyzing circuits. The Wiener-Kinchine theorem relates the autocorrelation function of a signal to the power spectrum of a signal; i.e., the autocorrelation function and the power spectrum of a signal form a Fourier transform pair [1]. The Fourier transform relationship between the autocorrelation function of the aperture distribution and the angular power spectrum establishes a theorem which is analogous to the Wiener-Kinchine theorem of circuits. The establishment of the analogous Wiener-Kinchine theorem offers an alternative method for solving aperture antenna problems [2,3].

In this paper, the normalized angular power spectrum representations for linear and circular aperture antennas in terms of their autocorrelation functions are derived. Results obtained by using these representations are compared with results determined through conventional methods [4].

THE AUTOCORRELATION FUNCTION AND ANGULAR POWER SPECTRUM FOR APERTURE ANTENNAS

The angular power spectrum for the linear aperture antenna is represented as [4]

$$|g(k\sin\theta)|^2 = \left| \int_{-L}^{L} f(x') e^{j\Phi(x')} e^{-jkx'\sin\theta} dx' \right|^2 \qquad (1)$$

where f(x') is the amplitude distribution, Φ(x') is the phase distribution, 2L is the aperture length, k is the wave number, and θ is the angle of measure normal to the aperture. Removing the magnitude symbol equation (1) becomes

$$|g(u)|^2 = \int_{-L}^{L}\int_{-L}^{L} f(x_1) f(x) e^{j\Phi(x_1)} e^{-j\Phi(x)} e^{-j(x_1-x)u} dx_1 dx \qquad (2)$$

where u= k sinθ . Letting $x_1 - x = \tau$,

$$|g(u)|^2 = \int_{x=-L}^{L}\int_{\tau=-L-x}^{L-x} f(x+\tau) f(x) e^{j[\Phi(x+\tau)-\Phi(x)]} e^{-ju\tau} d\tau\, dx \qquad (3)$$

Interchanging the order of integrations (see Fig. 1), equation (3) can be written as

$$|g(u)|^2 = \int_{\tau=0}^{2L} e^{-ju\tau} \int_{x=-L}^{L-\tau} f(x+\tau) f(x) e^{j[\Phi(x+\tau)-\Phi(x)]} dx\, d\tau$$

$$+ \int_{\tau=-2L}^{0} e^{-ju\tau} \int_{x=-L-\tau}^{L} f(x+\tau) f(x) e^{j[\Phi(x+\tau)-\Phi(x)]} dx\, d\tau \qquad (4)$$

In the second integral with $x+\tau=x$, equation (4) becomes

$$|g(u)|^2 = \int_{\tau=0}^{2L} e^{-ju\tau} \int_{x=-L}^{L-\tau} f(x+\tau) f(x) e^{j[\Phi(x+\tau)-\Phi(x)]} dx\, d\tau$$
$$+ \int_{\tau=-2L}^{0} e^{-ju\tau} \int_{x=-L}^{L+\tau} f(x) f(x-\tau) e^{j[\Phi(x)-\Phi(x-\tau)]} dx\, d\tau \qquad (5)$$

Define the aperture autocorrelation function as

$$\beta(\tau) = \frac{1}{\int_{-L}^{L} f^2 dx} \int_{x=-L}^{L-\tau} f(x+\tau) f(x)\, e^{j[\Phi(x+\tau)-\Phi(x)]} dx \qquad (6)$$

The angular power spectrum is now more compactly expressed as

$$|g(u)|^2 = \int_{-L}^{L} f^2 dx \int_{-\infty}^{\infty} Q(\tau) e^{-ju\tau} d\tau \qquad (7)$$

where

$$Q(\tau) = \begin{Bmatrix} \beta(\tau) & 0<\tau<2L \\ \beta^*(-\tau) & -2L<\tau<0 \\ 0 & |\tau|>2L \end{Bmatrix} \qquad (8)$$

and the * denotes complex conjugate. Normalizing each side of equation(7) by $\int f^2 dx$, the normalized angular power spectrum becomes

$$P_N(u) = \int_{-\infty}^{\infty} Q(\tau) e^{-ju\tau} d\tau \qquad (9)$$

Then, from the theory of Fourier integrals

$$Q(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} P_N(u) e^{ju\tau} du \qquad (10)$$

For the linear aperture antenna, this transform pair is the Wiener-Kinchine theorem which says simply the normalized angular power spectrum of the aperture distribution is the Fourier transform of its autocorrelation function.

The angular power spectrum for the circular aperture is represented as [4]

$$|g(\theta,\phi)|^2 = \left|\int\!\!\int f(\rho)\, e^{j\Phi(\rho)} e^{-j\bar{k}\cdot\bar{\rho}}\, dS\right|^2 \qquad (11)$$

with $\bar{k} = \hat{x}\, k\sin\theta\cos\phi + \hat{y}\, k\sin\theta\sin\phi$ and $\bar{\rho} = \rho[\hat{x}\cos\Phi' + \hat{y}\sin\Phi']$ where ρ and Φ' are the variables of integration and θ and ϕ designate the far-field point.

Expanding the right-hand side of equation (11), the angular power spectrum becomes

$$|g(\theta,\phi)|^2 = \int_{S_1}\!\int_{S_2} f(\bar{\rho}_1) f(\bar{\rho}_2)\, e^{j[\Phi(\bar{\rho}_1)-\Phi(\bar{\rho}_2)]}\, e^{-j\bar{k}\cdot[\bar{\rho}_1-\bar{\rho}_2]}\, dS_1\, dS_2 \qquad (12)$$

With $\bar{\rho}_1 - \bar{\rho}_2 = \bar{\tau}$, one has

$$|g(\theta,\phi)|^2 = \int_{S_\tau}\!\int_{S_2} f(\bar{\rho}_2+\bar{\tau}) f(\bar{\rho}_2)\, e^{j[\Phi(\bar{\rho}_2+\bar{\tau})-\Phi(\bar{\rho}_2)]}\, e^{-j\bar{k}\cdot\bar{\tau}}\, dS_2\, dS_\tau \qquad (13)$$

The magnitude of the vector $\bar{\tau}$ can vary from zero to the total diameter D of the aperture and its direction α with respect to some reference can vary from zero to 2π. The region common to the original aperture and the shifted aperture establishes the limits of integration for S_2 for fixed τ. Define the autocorrelation function as

$$\beta(\bar{\tau}) = \frac{1}{\int_S f^2\, dS} \int_{\text{Common Area}} f(\bar{\rho}_2+\bar{\tau}) f(\bar{\rho}_2)\, e^{j[\Phi(\bar{\rho}_2+\bar{\tau})-\Phi(\bar{\rho}_2)]}\, dS_2 \qquad (14)$$

Therefore, equation (13) can be written as

$$|g(\theta,\phi)|^2 = \int_S f^2(\rho)\,dS \int_{S\tau} \beta(\bar{\tau})\, e^{-j\bar{k}\cdot\bar{\tau}}\, d\tau \qquad (15)$$

Again normalizing with $\int f^2\,dS$, the normalized angular power spectrum becomes

$$P_N = \int_{S\tau} \beta(\bar{\tau})\, e^{-j\bar{k}\cdot\bar{\tau}}\, dS_\tau \qquad (16)$$

or explicitly,

$$P_N = \int_0^D \int_0^{2\pi} \beta(\tau,\alpha)\, e^{-jk\tau\sin\theta\cos(\theta-\alpha)}\, \tau\, d\tau\, d\alpha \qquad (17)$$

Aperture distributions which depend only on the radial variable (ρ) produce autocorrelation functions (β) independent of α. The α- integration yields

$$P_N(u) = 2\pi \int_0^\infty \beta(\tau)\, J_0(u\tau)\, \tau\, d\tau \qquad (18)$$

with $u = k\sin\theta$ and

$$\beta(\tau) = 0 \qquad \tau > D \qquad (19)$$

This is identified as the Fourier-Bessel transform, of order 0, of the autocorrelation function $\beta(\tau)$. The function $\beta(\tau)$, therefore, is expressible in terms of its transform as [5]

$$\beta(\tau) = \frac{1}{2\pi} \int_0^\infty u\, P_N(u)\, J_0(u\tau)\, du \qquad (20)$$

Equations (18) and (20) establish a Fourier transform pair which is analogous to the Wiener-Kinchine theorem for circuits.

APPLICATIONS

In aperture antennas such as reflectors, aperture distributions, in particular phase distributions, can be either deterministic or random. Deterministic errors are types which can be described analytically, thus, enabling one to compute the angular power spectrum in a straight forward manner. Random errors, on the other hand, one must resort to an average angular power spectrum.

The deterministic type phase error is considered first. For the linear aperture antenna, assume the amplitude is uniform and the phase is linearly tapered. The autocorrelation function becomes

$$Q(\tau) = \begin{Bmatrix} e^{-ja\tau}\dfrac{(2L-|\tau|)}{2L} & |\tau|<2L \\ 0 & |\tau|>2L \end{Bmatrix} \qquad (21)$$

From the convolution integral, the angular power spectrum can be written as

$$P_N(u) = \mathfrak{F}\left\{e^{-ja\tau}\right\} * \mathfrak{F}\left\{\frac{2L-|\tau|}{2L}\right\} \qquad (22)$$

or

$$P_N(u) = \delta(u-a) * 2L\frac{\sin^2 uL}{(uL)^2} \qquad (23)$$

where $\mathfrak{F}$ denotes the Fourier transform, δ the delta function, and the * means convolution. Thus,

$$P_N(u) = 2L\frac{\sin^2(u-a)L}{[(u-a)L]^2} \qquad (24)$$

which agrees with Silver [4] .

By way of illustration of the random phase type, consider the problem investigated first by Ruze[6] and later by Vu [7] . In these works, the phase distribution was assumed to be random, coming from a Gaussian population of rms error σ . The shape of the phase was also chosen as Gaussian with correlation length c. These random errors form a statistical problem. One can, therefore, only consider on the average what happens to the angular power spectrum. The average value of the linear and circular aperture antennas angular power spectrums are given, respectively, as

$$\langle P_{NL}\rangle = \int_{-\infty}^{\infty} \langle Q(\tau)\rangle e^{ju\tau}\, d\tau \tag{25}$$

$$\langle P_{Nc}\rangle = 2\pi \int_{0}^{\infty} \langle \beta(\tau)\rangle J_0(u\tau)\tau\, d\tau \tag{26}$$

where the symbol $\langle\rangle$ denotes the average value. Incorporating the statistical nature assumed in Ruze's and Vu's works, equations (25) and (26) become

$$\langle P_{NL}\rangle = \int_{-\infty}^{\infty} \left[e^{-\sigma^2\left(1-e^{-\frac{\tau^2}{c^2}}\right)}\right] Q'(\tau) e^{ju\tau}\, d\tau \tag{27}$$

$$\langle P_{Nc}\rangle = 2\pi \int_{0}^{\infty} \left[e^{-\sigma^2\left(1-e^{-\frac{\tau^2}{c^2}}\right)}\right] \beta'(\tau) J_0(u\tau)\tau\, d\tau \tag{28}$$

where

$$Q'(\tau) = \begin{cases} \dfrac{1}{\int_{-L}^{L} f^2 dx} \displaystyle\int_{-L}^{L-|\tau|} f(x) f(x+|\tau|)\, dx & |\tau| < 2L \\ 0 & |\tau| > 2L \end{cases} \tag{29}$$

$$\beta'(\tau) = \frac{1}{\int_S f^2 dS} \int_{\substack{\text{Common} \\ \text{Area}}} f(\rho_2) f(\rho_2+\tau)\, dS_2 \tag{30}$$

which are the amplitude autocorrelation functions for the linear and circular aperture antennas, respectively.

By expressing the bracketed terms in the integrands of equations (27) and (28) in a series, the normalized angular power spectrums for the linear and circular apertures can be represented in a more tractible form for evaluation. Thus,

$$\langle P_{NL} \rangle = e^{-\sigma^2} \int_{-\infty}^{\infty} Q'(\tau) e^{ju\tau} d\tau + e^{-\sigma^2} \int_{-\infty}^{\infty} \left[\sum_{n=1}^{\infty} \frac{\sigma^{2n}}{n!} e^{-\frac{n\tau^2}{c^2}} \right] Q'(\tau) e^{ju\tau} d\tau \qquad (32)$$

$$\langle P_{NC} \rangle = 2\pi e^{-\sigma^2} \int_{0}^{\infty} \beta'(\tau) J_0(u\tau) \tau d\tau + 2\pi e^{-\sigma^2} \int_{0}^{\infty} \left[\sum_{n=1}^{\infty} \frac{\sigma^{2n}}{n!} e^{-\frac{n\tau^2}{c^2}} \right] \beta'(\tau) J_0(u\tau) \tau d\tau \qquad (33)$$

The first term in each of equations (32) and (33) is the product of the error-free normalized angular power spectrum and an exponential factor which depends on the rms phase error. The second term in each, the so-called "scattered radiation," is an infinite series which is a function of the rms phase error, the amplitude autocorrelation function, and correlation length c. For complete discussions of the effects these parameters have on the normalized angular power spectrums refer to papers by Ruze and Vu.

For the error-free case ($\sigma = 0$), equations (32) and (33) become, respectively

$$\langle P_{NL} \rangle = P_{NL} = \int_{-\infty}^{\infty} Q'(\tau) e^{ju\tau} d\tau \qquad (34)$$

$$\langle P_{NC} \rangle = P_{NC} = 2\pi \int_{0}^{\infty} \beta'(\tau) J_0(u\tau) \tau d\tau \qquad (35)$$

which for specific aperture distributions can be shown to agree with the

normalized angular power spectrums obtained through conventional methods. As an example, consider the case of uniformly illuminated apertures with no phase error, equations (34) and (35) become

$$P_{NL} = \frac{1}{2L}\int_{-2L}^{2L} (2L - |\tau|)e^{ju\tau}d\tau \qquad (36)$$

$$P_{NC} = \frac{8}{D^2}\int_{0}^{D}\left[\frac{D^2}{2}\cos^{-1}\frac{\tau}{D} - \tau\sqrt{\left(\frac{D}{2}\right)^2-\left(\frac{\tau}{2}\right)^2}\right] J_0(u\tau)\tau d\tau \qquad (37)$$

Performing the indicated integrations, the normalized angular power spectrums for uniformly illuminated linear and circular apertures with no phase error become

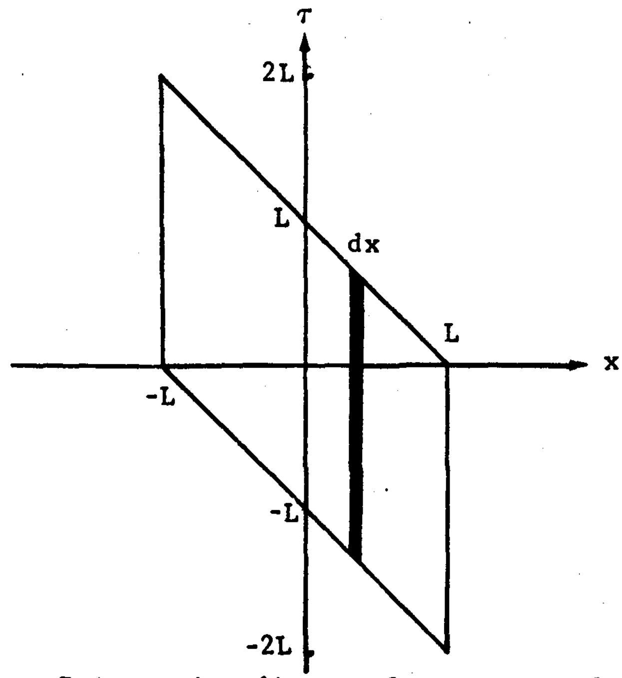
$$P_{NL} = 2L\left(\frac{\sin uL}{uL}\right)^2 \qquad (38)$$

$$P_{NC} = \pi D^2\left(\frac{J_1\left(\frac{uD}{2}\right)}{\frac{uD}{2}}\right)^2 \qquad (39)$$

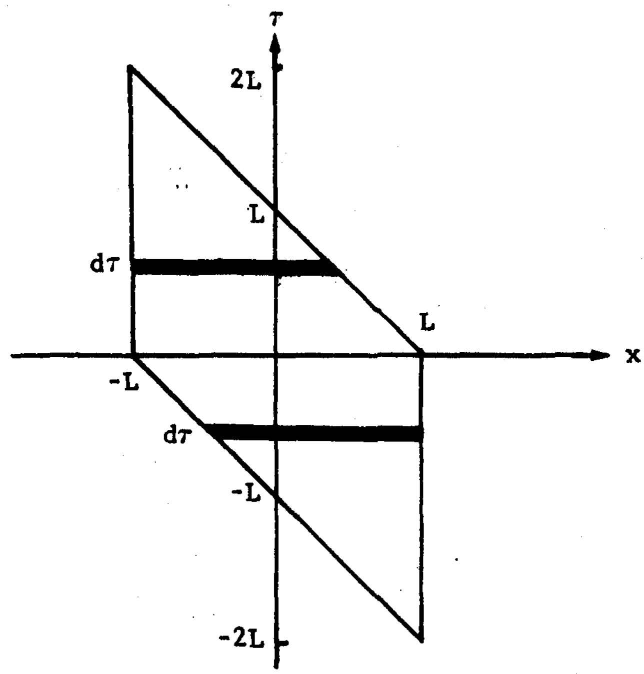
which are in agreement with results evaluated by conventional techniques[4].

CONCLUDING REMARKS

Normalized angular power spectrum representations have been derived for linear and circular aperture antennas in terms of their autocorrelation functions;i.e., Fourier transform pairs relate these functions. Both deterministic and random type aperture distributions can be used in these relationships.



Integration diagram for equation 3



Integration diagram for equation 4

Figure 1. Integration Diagrams

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16. Abstract

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